

## 2-IRREDUCIBLE AND STRONGLY 2-IRREDUCIBLE IDEALS OF COMMUTATIVE RINGS

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**ABSTRACT.** An ideal  $I$  of a commutative ring  $R$  is said to be *irreducible* if it cannot be written as the intersection of two larger ideals. A proper ideal  $I$  of a ring  $R$  is said to be *strongly irreducible* if for each ideals  $J, K$  of  $R$ ,  $J \cap K \subseteq I$  implies that  $J \subseteq I$  or  $K \subseteq I$ . In this paper, we introduce the concepts of 2-irreducible and strongly 2-irreducible ideals which are generalizations of irreducible and strongly irreducible ideals, respectively. We say that a proper ideal  $I$  of a ring  $R$  is *2-irreducible* if for each ideals  $J, K$  and  $L$  of  $R$ ,  $I = J \cap K \cap L$  implies that either  $I = J \cap K$  or  $I = J \cap L$  or  $I = K \cap L$ . A proper ideal  $I$  of a ring  $R$  is called *strongly 2-irreducible* if for each ideals  $J, K$  and  $L$  of  $R$ ,  $J \cap K \cap L \subseteq I$  implies that either  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ .

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### 1. Introduction

Throughout this paper all rings are commutative with a nonzero identity. Recall that an ideal  $I$  of a commutative ring  $R$  is *irreducible* if  $I = J \cap K$  for ideals  $J$  and  $K$  of  $R$  implies that either  $I = J$  or  $I = K$ . A proper ideal  $I$  of a ring  $R$  is said to be *strongly irreducible* if for each ideals  $J, K$  of  $R$ ,  $J \cap K \subseteq I$  implies that  $J \subseteq I$  or  $K \subseteq I$  (see [3], [8]). Obviously a proper ideal  $I$  of a ring  $R$  is strongly irreducible if and only if for each  $x, y \in R$ ,  $Rx \cap Ry \subseteq I$  implies that  $x \in I$  or  $y \in I$ . It is easy to see that any strongly irreducible ideal is an irreducible ideal. Now, we recall some definitions which are the motivation of our work. Badawi in [4] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal  $I$  of  $R$  to be a *2-absorbing ideal* of  $R$  if whenever  $a, b, c \in R$  and  $abc \in I$ , then  $ab \in I$  or  $ac \in I$  or  $bc \in I$ . It is shown that a proper ideal  $I$  of  $R$  is a 2-absorbing ideal if and only if whenever  $I_1 I_2 I_3 \subseteq I$  for some ideals  $I_1, I_2, I_3$  of  $R$ , then  $I_1 I_2 \subseteq I$  or  $I_1 I_3 \subseteq I$  or  $I_2 I_3 \subseteq I$ . In [18], Yousefian Darani and Puczyłowski studied the concept of 2-absorbing commutative semigroups. Anderson and Badawi [2] generalized the concept of 2-absorbing ideals to  $n$ -absorbing ideals. According to their definition, a proper ideal  $I$  of  $R$  is called an  *$n$ -absorbing* (resp. *strongly  $n$ -absorbing*)

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ideal if whenever  $a_1 \cdots a_{n+1} \in I$  for  $a_1, \dots, a_{n+1} \in R$  (resp.  $I_1 \cdots I_{n+1} \subseteq I$  for ideals  $I_1, \dots, I_{n+1}$  of  $R$ ), then there are  $n$  of the  $a_i$ 's (resp.  $n$  of the  $I_i$ 's) whose product is in  $I$ . Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly  $n$ -absorbing ideal of  $R$  is also an  $n$ -absorbing ideal of  $R$ . The concept of 2-absorbing primary ideals, a generalization of primary ideals was introduced and investigated in [5]. A proper ideal  $I$  of a commutative ring  $R$  is called a *2-absorbing primary ideal* if whenever  $a, b, c \in R$  and  $abc \in I$ , then either  $ab \in I$  or  $ac \in \sqrt{I}$  or  $bc \in \sqrt{I}$ . We refer the readers to [6] for a specific kind of 2-absorbing ideals and to [14], [19], [20] for the module version of the above definitions. We define an ideal  $I$  of a ring  $R$  to be *2-irreducible* if whenever  $I = J \cap K \cap L$  for ideals  $I, J$  and  $K$  of  $R$ , then either  $I = J \cap K$  or  $I = J \cap L$  or  $I = K \cap L$ . Obviously, any irreducible ideal is a 2-irreducible ideal. Also, we say that a proper ideal  $I$  of a ring  $R$  is called *strongly 2-irreducible* if for each ideals  $J, K$  and  $L$  of  $R$ ,  $J \cap K \cap L \subseteq I$  implies that  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ . Clearly, any strongly irreducible ideal is a strongly 2-irreducible ideal. In [17], [16] we can find the notion of 2-irreducible preradicals and its dual, the notion of co-2-irreducible preradicals. We call a proper ideal  $I$  of a ring  $R$  *singly strongly 2-irreducible* if for each  $x, y, z \in R$ ,  $Rx \cap Ry \cap Rz \subseteq I$  implies that  $Rx \cap Ry \subseteq I$  or  $Rx \cap Rz \subseteq I$  or  $Ry \cap Rz \subseteq I$ . It is trivial that any strongly 2-irreducible ideal is a singly strongly 2-irreducible ideal. A ring  $R$  is said to be an *arithmetical ring*, if for each ideals  $I, J$  and  $K$  of  $R$ ,  $(I + J) \cap K = (I \cap K) + (J \cap K)$ . This condition is equivalent to the condition that for each ideals  $I, J$  and  $K$  of  $R$ ,  $(I \cap J) + K = (I + K) \cap (J + K)$ , see [10]. In this paper we prove that, a nonzero ideal  $I$  of a principal ideal domain  $R$  is 2-irreducible if and only if  $I$  is strongly 2-irreducible if and only if  $I$  is 2-absorbing primary. It is shown that a proper ideal  $I$  of a ring  $R$  is strongly 2-irreducible if and only if for each  $x, y, z \in R$ ,  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq I$  implies that  $(Rx + Ry) \cap (Rx + Rz) \subseteq I$  or  $(Rx + Ry) \cap (Ry + Rz) \subseteq I$  or  $(Rx + Rz) \cap (Ry + Rz) \subseteq I$ . A proper ideal  $I$  of a von Neumann regular ring  $R$  is 2-irreducible if and only if  $I$  is 2-absorbing if and only if for every idempotent elements  $e_1, e_2, e_3$  of  $R$ ,  $e_1 e_2 e_3 \in I$  implies that either  $e_1 e_2 \in I$  or  $e_1 e_3 \in I$  or  $e_2 e_3 \in I$ . If  $I$  is a 2-irreducible ideal of a Noetherian ring  $R$ , then  $I$  is a 2-absorbing primary ideal of  $R$ . Let  $R = R_1 \times R_2$ , where  $R_1$  and  $R_2$  are commutative rings with  $1 \neq 0$ . It is shown that a proper ideal  $J$  of  $R$  is a strongly 2-irreducible ideal of  $R$  if and only if either  $J = I_1 \times R_2$  for some strongly 2-irreducible ideal  $I_1$  of  $R_1$  or  $J = R_1 \times I_2$  for some strongly 2-irreducible ideal  $I_2$  of  $R_2$  or  $J = I_1 \times I_2$  for some strongly irreducible ideal  $I_1$  of  $R_1$  and some strongly irreducible ideal  $I_2$  of  $R_2$ . A proper ideal  $I$  of a unique factorization domain  $R$  is singly strongly 2-irreducible if and only if  $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$ , where  $p_i$ 's are distinct prime elements of  $R$  and  $n_i$ 's are natural numbers, implies that  $p_r^{n_r} p_s^{n_s} \in I$ , for some  $1 \leq r, s \leq k$ .

## 2. Basic properties of 2-irreducible and strongly 2-irreducible ideals

It is important to notice that when  $R$  is a domain, then  $R$  is an arithmetical ring if and only if  $R$  is a Prüfer domain. In particular, every Dedekind domain is an arithmetical domain.

**Theorem 2.1.** *Let  $R$  be a Dedekind domain and  $I$  be a nonzero proper ideal of  $R$ . The following conditions are equivalent:*

- (1)  $I$  is a strongly irreducible ideal;
- (2)  $I$  is an irreducible ideal;
- (3)  $I$  is a primary ideal;
- (4)  $I = Rp^n$  for some prime (irreducible) element  $p$  of  $R$  and some natural number  $n$ .

*Proof.* See [8, Lemma 2.2(3)] and [13, p. 130, Exercise 36].  $\square$

We recall from [1] that an integral domain  $R$  is called a *GCD-domain* if any two nonzero elements of  $R$  have a greatest common divisor (*GCD*), equivalently, any two nonzero elements of  $R$  have a least common multiple (*LCM*). Unique factorization domains (*UFD*'s) are well-known examples of *GCD*-domains. Let  $R$  be a *GCD*-domain. The least common multiple of elements  $x, y$  of  $R$  is denoted by  $[x, y]$ . Notice that for every elements  $x, y \in R$ ,  $Rx \cap Ry = R[x, y]$ . Moreover, for every elements  $x, y, z$  of  $R$ , we have  $[[x, y], z] = [x, [y, z]]$ . So we denote  $[[x, y], z]$  simply by  $[x, y, z]$ .

Recall that every principal ideal domain (*PID*) is a Dedekind domain.

**Theorem 2.2.** *Let  $R$  be a *PID* and  $I$  be a nonzero proper ideal of  $R$ . The following conditions are equivalent:*

- (1)  $I$  is a 2-irreducible ideal;
- (2)  $I$  is a 2-absorbing primary ideal;
- (3) Either  $I = Rp^k$  for some prime (irreducible) element  $p$  of  $R$  and some natural number  $n$ , or  $I = R(p_1^n p_2^m)$  for some distinct prime (irreducible) elements  $p_1, p_2$  of  $R$  and some natural numbers  $n, m$ .

*Proof.* (2) $\Leftrightarrow$ (3) See [5, Corollary 2.12].

(1) $\Rightarrow$ (3) Assume that  $I = Ra$  where  $0 \neq a \in R$ . Let  $a = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$  be a prime decomposition for  $a$ . We show that either  $k = 1$  or  $k = 2$ . Suppose that  $k > 2$ . By [9, p. 141, Exercise 5], we have that  $I = Rp_1^{n_1} \cap Rp_2^{n_2} \cap \cdots \cap Rp_k^{n_k}$ . Now, since  $I$  is 2-irreducible, there exist  $1 \leq i, j \leq k$  such that  $I = Rp_i^{n_i} \cap Rp_j^{n_j}$ , say  $i = 1, j = 2$ . Therefore we have  $I = Rp_1^{n_1} \cap Rp_2^{n_2} \subseteq Rp_3^{n_3}$ , which is a contradiction.

(3) $\Rightarrow$ (1) If  $I = Rp^k$  for some prime element  $p$  of  $R$  and some natural number  $n$ , then  $I$  is irreducible, by Theorem 2.1, and so  $I$  is 2-irreducible. Therefore, assume that  $I = R(p_1^n p_2^m)$  for some distinct prime elements  $p_1, p_2$  of  $R$  and some natural numbers  $n, m$ . Let  $I = Ra \cap Rb \cap Rc$  for some elements  $a, b$

and  $c$  of  $R$ . Then  $a$ ,  $b$  and  $c$  divide  $p_1^n p_2^m$ , and so  $a = p_1^{\alpha_1} p_2^{\alpha_2}$ ,  $b = p_1^{\beta_1} p_2^{\beta_2}$  and  $c = p_1^{\gamma_1} p_2^{\gamma_2}$  where  $\alpha_i, \beta_i, \gamma_i$  are some nonnegative integers. On the other hand  $I = Ra \cap Rb \cap Rc = R[a, b, c] = R(p_1^\delta p_2^\varepsilon)$  in which  $\delta = \max\{\alpha_1, \beta_1, \gamma_1\}$  and  $\varepsilon = \max\{\alpha_2, \beta_2, \gamma_2\}$ . We can assume without loss of generality that  $\delta = \alpha_1$  and  $\varepsilon = \beta_2$ . So  $I = R(p_1^{\alpha_1} p_2^{\beta_2}) = Ra \cap Rb$ . Consequently,  $I$  is 2-irreducible.  $\square$

A commutative ring  $R$  is called a *von Neumann regular ring* (or an *absolutely flat ring*) if for any  $a \in R$  there exists an  $x \in R$  with  $a^2 x = a$ , equivalently,  $I = I^2$  for every ideal  $I$  of  $R$ .

*Remark 2.3.* Notice that a commutative ring  $R$  is a von Neumann regular ring if and only if  $IJ = I \cap J$  for any ideals  $I, J$  of  $R$ , by [11, Lemma 1.2]. Therefore over a commutative von Neumann regular ring the two concepts of strongly 2-irreducible ideals and of 2-absorbing ideals are coincide.

**Theorem 2.4.** *Let  $I$  be a proper ideal of a ring  $R$ . Then the following conditions are equivalent:*

- (1)  *$I$  is strongly 2-irreducible;*
- (2) *For every elements  $x, y, z$  of  $R$ ,  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq I$  implies that  $(Rx + Ry) \cap (Rx + Rz) \subseteq I$  or  $(Rx + Ry) \cap (Ry + Rz) \subseteq I$  or  $(Rx + Rz) \cap (Ry + Rz) \subseteq I$ .*

*Proof.* (1) $\Rightarrow$ (2) There is nothing to prove.

(2) $\Rightarrow$ (1) Suppose that  $J, K$  and  $L$  are ideals of  $R$  such that neither  $J \cap K \subseteq I$  nor  $J \cap L \subseteq I$  nor  $K \cap L \subseteq I$ . Then there exist elements  $x, y$  and  $z$  of  $R$  such that  $x \in (J \cap K) \setminus I$  and  $y \in (J \cap L) \setminus I$  and  $z \in (K \cap L) \setminus I$ . On the other hand  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq (Rx + Ry) \subseteq J$ ,  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq (Rx + Rz) \subseteq K$  and  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq (Ry + Rz) \subseteq L$ . Hence  $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq I$ , and so by hypothesis either  $(Rx + Ry) \cap (Rx + Rz) \subseteq I$  or  $(Rx + Ry) \cap (Ry + Rz) \subseteq I$  or  $(Rx + Rz) \cap (Ry + Rz) \subseteq I$ . Therefore, either  $x \in I$  or  $y \in I$  or  $z \in I$ , which any of these cases has a contradiction. Consequently  $I$  is strongly 2-irreducible.  $\square$

A ring  $R$  is called a *Bézout ring* if every finitely generated ideal of  $R$  is principal.

As an immediate consequence of Theorem 2.4 we have the next result:

**Corollary 2.5.** *Let  $I$  be a proper ideal of a Bézout ring  $R$ . Then the following conditions are equivalent:*

- (1)  *$I$  is strongly 2-irreducible;*
- (2)  *$I$  is singly strongly 2-irreducible;*

Now we can state the following open problem.

**Problem 2.6.** Let  $I$  be a singly strongly 2-irreducible ideal of a ring  $R$ . Is  $I$  a strongly 2-irreducible ideal of  $R$ ?

**Proposition 2.7.** *Let  $R$  be a ring. If  $I$  is a strongly 2-irreducible ideal of  $R$ , then  $I$  is a 2-irreducible ideal of  $R$ .*

*Proof.* Suppose that  $I$  is a strongly 2-irreducible ideal of  $R$ . Let  $J$ ,  $K$  and  $L$  be ideals of  $R$  such that  $I = J \cap K \cap L$ . Since  $J \cap K \cap L \subseteq I$ , then either  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ . On the other hand  $I \subseteq J \cap K$  and  $I \subseteq J \cap L$  and  $I \subseteq K \cap L$ . Consequently, either  $I = J \cap K$  or  $I = J \cap L$  or  $I = K \cap L$ . Therefore  $I$  is 2-irreducible.  $\square$

*Remark 2.8.* It is easy to check that the zero ideal  $I = \{0\}$  of a ring  $R$  is 2-irreducible if and only if  $I$  is strongly 2-irreducible.

**Proposition 2.9.** *Let  $I$  be a proper ideal of an arithmetical ring  $R$ . The following conditions are equivalent:*

- (1)  $I$  is a 2-irreducible ideal of  $R$ ;
- (2)  $I$  is a strongly 2-irreducible ideal of  $R$ ;
- (3) For every ideals  $I_1, I_2$  and  $I_3$  of  $R$  with  $I \subseteq I_1, I_1 \cap I_2 \cap I_3 \subseteq I$  implies that  $I_1 \cap I_2 \subseteq I$  or  $I_1 \cap I_3 \subseteq I$  or  $I_2 \cap I_3 \subseteq I$ .

*Proof.* (1) $\Rightarrow$ (2) Assume that  $J, K$  and  $L$  are ideals of  $R$  such that  $J \cap K \cap L \subseteq I$ . Therefore  $I = I + (J \cap K \cap L) = (I + J) \cap (I + K) \cap (I + L)$ , since  $R$  is an arithmetical ring. So either  $I = (I + J) \cap (I + K)$  or  $I = (I + J) \cap (I + L)$  or  $I = (I + K) \cap (I + L)$ , and thus either  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ . Hence  $I$  is a strongly 2-irreducible ideal.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (2) Let  $J, K$  and  $L$  be ideals of  $R$  such that  $J \cap K \cap L \subseteq I$ . Set  $I_1 := J + I, I_2 := K$  and  $I_3 := L$ . Since  $R$  is an arithmetical ring, then  $I_1 \cap I_2 \cap I_3 = (J + I) \cap K \cap L = (J \cap K \cap L) + (I \cap K \cap L) \subseteq I$ . Hence either  $I_1 \cap I_2 \subseteq I$  or  $I_1 \cap I_3 \subseteq I$  or  $I_2 \cap I_3 \subseteq I$  which imply that either  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ , respectively. Consequently,  $I$  is a strongly 2-irreducible ideal of  $R$ .

(2) $\Rightarrow$ (1) By Proposition 2.7.  $\square$

As an immediate consequence of Theorem 2.2 and Proposition 2.9 we have the next result.

**Corollary 2.10.** *Let  $R$  be a PID and  $I$  be a nonzero proper ideal of  $R$ . The following conditions are equivalent:*

- (1)  $I$  is a strongly 2-irreducible ideal;
- (2)  $I$  is a 2-irreducible ideal;
- (3)  $I$  is a 2-absorbing primary ideal;
- (4) Either  $I = Rp^k$  for some prime (irreducible) element  $p$  of  $R$  and some natural number  $n$ , or  $I = R(p_1^n p_2^m)$  for some distinct prime (irreducible) elements  $p_1, p_2$  of  $R$  and some natural numbers  $n, m$ .

The following example shows that the concepts of strongly irreducible (irreducible) ideals and of strongly 2-irreducible (2-irreducible) ideals are different in general.

**Example 2.11.** Consider the ideal  $6\mathbb{Z}$  of the ring  $\mathbb{Z}$ . By Corollary 2.10,  $6\mathbb{Z} = (2,3)\mathbb{Z}$  is a strongly 2-irreducible (a 2-irreducible) ideal of  $\mathbb{Z}$ . But, Theorem 2.1 says that  $6\mathbb{Z}$  is not a strongly irreducible (an irreducible) ideal of  $\mathbb{Z}$ .

It is well known that every von Neumann regular ring is a Bézout ring. By [10, p. 119], every Bézout ring is an arithmetical ring.

**Corollary 2.12.** *Let  $I$  be a proper ideal of a von Neumann regular ring  $R$ . The following conditions are equivalent:*

- (1)  $I$  is a 2-absorbing ideal of  $R$ ;
- (2)  $I$  is a 2-irreducible ideal of  $R$ ;
- (3)  $I$  is a strongly 2-irreducible ideal of  $R$ ;
- (4)  $I$  is a singly strongly 2-irreducible of  $R$ ;
- (5) For every idempotent elements  $e_1, e_2, e_3$  of  $R$ ,  $e_1e_2e_3 \in I$  implies that either  $e_1e_2 \in I$  or  $e_1e_3 \in I$  or  $e_2e_3 \in I$ .

*Proof.* (1) $\Leftrightarrow$ (3) By Remark 2.3.

(2) $\Leftrightarrow$ (3) By Proposition 2.9.

(3) $\Leftrightarrow$ (4) By Corollary 2.5.

(1) $\Rightarrow$ (5) is evident.

(5) $\Rightarrow$ (3) The proof follows from Theorem 2.4 and the fact that any finitely generated ideal of a von Neumann regular ring  $R$  is generated by an idempotent element.  $\square$

**Proposition 2.13.** *Let  $I_1, I_2$  be strongly irreducible ideals of a ring  $R$ . Then  $I_1 \cap I_2$  is a strongly 2-irreducible ideal of  $R$ .*

*Proof.* Straightforward.  $\square$

**Theorem 2.14.** *Let  $R$  be a Noetherian ring. If  $I$  is a 2-irreducible ideal of  $R$ , then either  $I$  is irreducible or  $I$  is the intersection of exactly two irreducible ideals. The converse is true when  $R$  is also arithmetical.*

*Proof.* Assume that  $I$  is 2-irreducible. By [15, Proposition 4.33],  $I$  can be written as a finite irredundant irreducible decomposition  $I = I_1 \cap I_2 \cap \cdots \cap I_k$ . We show that either  $k = 1$  or  $k = 2$ . If  $k > 3$ , then since  $I$  is 2-irreducible,  $I = I_i \cap I_j$  for some  $1 \leq i, j \leq k$ , say  $i = 1$  and  $j = 2$ . Therefore  $I_1 \cap I_2 \subseteq I_3$ , which is a contradiction. For the second atatement, let  $R$  be arithmetical, and  $I$  be the intersection of two irreducible ideals. Since  $R$  is arithmetical, every irreducible ideal is strongly irreducible, [8, Lemma 2.2(3)]. Now, apply Proposition 2.13 to see that  $I$  is strongly 2-irreducible, and so  $I$  is 2-irreducible.  $\square$

**Corollary 2.15.** *Let  $R$  be a Noetherian ring and  $I$  be a proper ideal of  $R$ . If  $I$  is 2-irreducible, then  $I$  is a 2-absorbing primary ideal of  $R$ .*

*Proof.* Assume that  $I$  is 2-irreducible. By the fact that every irreducible ideal of a Noetherian ring is primary and regarding Theorem 2.14, we have

either  $I$  is a primary ideal or is the intersection of two primary ideals. It is clear that every primary ideal is 2-absorbing primary, also the intersection of two primary ideals is a 2-absorbing primary ideal, by [5, Theorem 2.4].  $\square$

**Proposition 2.16.** *Let  $R$  be a ring, and let  $P_1$ ,  $P_2$  and  $P_3$  be pairwise comaximal prime ideals of  $R$ . Then  $P_1P_2P_3$  is not a 2-irreducible ideal.*

*Proof.* The proof is easy.  $\square$

**Corollary 2.17.** *If  $R$  is a ring such that every proper ideal of  $R$  is 2-irreducible, then  $R$  has at most two maximal ideals.*

**Theorem 2.18.** *Let  $I$  be a radical ideal of a ring  $R$ , i.e.,  $I = \sqrt{I}$ . The following conditions are equivalent:*

- (1)  $I$  is strongly 2-irreducible;
- (2)  $I$  is 2-absorbing;
- (3)  $I$  is 2-absorbing primary;
- (4)  $I$  is either a prime ideal of  $R$  or is an intersection of exactly two prime ideals of  $R$ .

*Proof.* (1) $\Rightarrow$ (2) Assume that  $I$  is strongly 2-irreducible. Let  $J$ ,  $K$  and  $L$  be ideals of  $R$  such that  $JKL \subseteq I$ . Then  $J \cap K \cap L \subseteq \sqrt{J \cap K \cap L} \subseteq \sqrt{I} = I$ . So, either  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ . Hence either  $JK \subseteq I$  or  $JL \subseteq I$  or  $KL \subseteq I$ . Consequently  $I$  is 2-absorbing.

(2) $\Leftrightarrow$ (3) is obvious.

(2) $\Rightarrow$ (4) If  $I$  is a 2-absorbing ideal, then either  $\sqrt{I}$  is a prime ideal or is an intersection of exactly two prime ideals, [4, Theorem 2.4]. Now, we prove the claim by assumption that  $I = \sqrt{I}$ .

(4) $\Rightarrow$ (1) By Proposition 2.13.  $\square$

**Theorem 2.19.** *Let  $f : R \rightarrow S$  be a surjective homomorphism of commutative rings, and let  $I$  be an ideal of  $R$  containing  $\text{Ker}(f)$ . Then,*

- (1) *If  $I$  is a strongly 2-irreducible ideal of  $R$ , then  $I^e$  is a strongly 2-irreducible ideal of  $S$ .*
- (2)  *$I$  is a 2-irreducible ideal of  $R$  if and only if  $I^e$  is a 2-irreducible ideal of  $S$ .*

*Proof.* Since  $f$  is surjective,  $J^{ce} = J$  for every ideal  $J$  of  $S$ . Moreover,  $(K \cap L)^e = K^e \cap L^e$  and  $K^{ec} = K$  for every ideals  $K$ ,  $L$  of  $R$  which contain  $\text{Ker}(f)$ .

(1) Suppose that  $I$  is a strongly 2-irreducible ideal of  $R$ . If  $I^e = S$ , then  $I = I^{ec} = R$ , which is a contradiction. Let  $J_1$ ,  $J_2$  and  $J_3$  be ideals of  $S$  such that  $J_1 \cap J_2 \cap J_3 \subseteq I^e$ . Therefore  $J_1^c \cap J_2^c \cap J_3^c \subseteq I^{ec} = I$ . So, either  $J_1^c \cap J_2^c \subseteq I$  or  $J_1^c \cap J_3^c \subseteq I$  or  $J_2^c \cap J_3^c \subseteq I$ . Without loss of generality, we may assume that  $J_1^c \cap J_2^c \subseteq I$ . So,  $J_1 \cap J_2 = (J_1 \cap J_2)^{ce} \subseteq I^e$ . Hence  $I^e$  is strongly 2-irreducible.

(2) The necessity is similar to part (1). Conversely, let  $I^e$  be a strongly 2-irreducible ideal of  $S$ , and let  $I_1$ ,  $I_2$  and  $I_3$  be ideals of  $R$  such that



$I = I_1 \cap I_2 \cap I_3$ . Then  $I^e = I_1^e \cap I_2^e \cap I_3^e$ . Hence, either  $I^e = I_1^e \cap I_2^e$  or  $I^e = I_1^e \cap I_3^e$  or  $I^e = I_2^e \cap I_3^e$ . We may assume that  $I^e = I_1^e \cap I_2^e$ . Therefore,  $I = I^{ec} = I_1^{ec} \cap I_2^{ec} = I_1 \cap I_2$ . Consequently,  $I$  is strongly 2-irreducible.  $\square$

**Corollary 2.20.** *Let  $f : R \rightarrow S$  be a surjective homomorphism of commutative rings. There is a one-to-one correspondence between the 2-irreducible ideals of  $R$  which contain  $\text{Ker}(f)$  and 2-irreducible ideals of  $S$ .*

Recall that a ring  $R$  is called a *Laskerian ring* if every proper ideal of  $R$  has a primary decomposition. Noetherian rings are some examples of Laskerian rings.

Let  $S$  be a multiplicatively closed subset of a ring  $R$ . In the next theorem, consider the natural homomorphism  $f : R \rightarrow S^{-1}R$  defined by  $f(x) = x/1$ .

**Theorem 2.21.** *Let  $I$  be a proper ideal of a ring  $R$  and  $S$  be a multiplicatively closed set in  $R$ .*

- (1) *If  $I$  is a strongly 2-irreducible ideal of  $S^{-1}R$ , then  $I^c$  is a strongly 2-irreducible ideal of  $R$ .*
- (2) *If  $I$  is a primary strongly 2-irreducible ideal of  $R$  such that  $I \cap S = \emptyset$ , then  $I^e$  is a strongly 2-irreducible ideal of  $S^{-1}R$ .*
- (3) *If  $I$  is a primary ideal of  $R$  such that  $I^e$  is a strongly 2-irreducible ideal of  $S^{-1}R$ , then  $I$  is a strongly 2-irreducible ideal of  $R$ .*
- (4) *If  $R'$  is a faithfully flat extension ring of  $R$  and if  $IR'$  is a strongly 2-irreducible ideal of  $R'$ , then  $I$  is a strongly 2-irreducible ideal of  $R$ .*
- (5) *If  $I$  is strongly 2-irreducible and  $H$  is an ideal of  $R$  such that  $H \subseteq I$ , then  $I/H$  is a strongly 2-irreducible ideal of  $R/H$ .*
- (6) *If  $R$  is a Laskerian ring, then every strongly 2-irreducible ideal is either a primary ideal or is the intersection of two primary ideals.*

*Proof.* (1) Assume that  $I$  is a strongly 2-irreducible ideal of  $S^{-1}R$ . Let  $J, K$  and  $L$  be ideals of  $R$  such that  $J \cap K \cap L \subseteq I^c$ . Then  $J^e \cap K^e \cap L^e \subseteq I^{ce} = I$ . Hence either  $J^e \cap K^e \subseteq I$  or  $J^e \cap L^e \subseteq I$  or  $K^e \cap L^e \subseteq I$  since  $I$  is strongly 2-irreducible. Therefore either  $J \cap K \subseteq I^c$  or  $J \cap L \subseteq I^c$  or  $K \cap L \subseteq I^c$ . Consequently  $I^c$  is a strongly 2-irreducible ideal of  $R$ .

(2) Suppose that  $I$  is a primary strongly 2-irreducible ideal such that  $I \cap S = \emptyset$ . Let  $J, K$  and  $L$  be ideals of  $S^{-1}R$  such that  $J \cap K \cap L \subseteq I^e$ . Since  $I$  is a primary ideal, then  $J^c \cap K^c \cap L^c \subseteq I^{ec} = I$ . Thus  $J^c \cap K^c \subseteq I$  or  $J^c \cap L^c \subseteq I$  or  $K^c \cap L^c \subseteq I$ . Hence  $J \cap K \subseteq I^e$  or  $J \cap L \subseteq I^e$  or  $K \cap L \subseteq I^e$ .

(3) Let  $I$  be a primary ideal of  $R$ , and let  $I^e$  be a strongly 2-irreducible ideal of  $S^{-1}R$ . By part (1),  $I^{ec}$  is strongly 2-irreducible. Since  $I$  is primary, we have  $I^{ec} = I$ , and thus we are done.

(4) Let  $J, K$  and  $L$  be ideals of  $R$  such that  $J \cap K \cap L \subseteq I$ . Thus  $JR' \cap KR' \cap LR' = (J \cap K \cap L)R' \subseteq IR'$ , by [7, Lemma 9.9]. Since  $IR'$  is strongly 2-irreducible, then either  $JR' \cap KR' \subseteq IR'$  or  $JR' \cap LR' \subseteq IR'$  or  $KR' \cap LR' \subseteq IR'$ . Without loss of generality, assume that  $JR' \cap KR' \subseteq IR'$ . So,  $(JR' \cap R) \cap (KR' \cap R) \subseteq IR' \cap R$ . Hence  $J \cap K \subseteq I$ , by [12, Theorem 4.74]. Consequently  $I$  is strongly 2-irreducible.



- (5) Let  $J$ ,  $K$  and  $L$  be ideals of  $R$  containing  $H$  such that  $(J/H) \cap (K/H) \cap (L/H) \subseteq I/H$ . Hence  $J \cap K \cap L \subseteq I$ . Therefore, either  $J \cap K \subseteq I$  or  $J \cap L \subseteq I$  or  $K \cap L \subseteq I$ . Thus,  $(J/H) \cap (K/H) \subseteq I/H$  or  $(J/H) \cap (L/H) \subseteq I/H$  or  $(K/H) \cap (L/H) \subseteq I/H$ . Consequently,  $I/H$  is strongly 2-irreducible.
- (6) Let  $I$  be a strongly 2-irreducible ideal and  $\cap_{i=1}^n Q_i$  be a primary decomposition of  $I$ . Since  $\cap_{i=1}^n Q_i \subseteq I$ , then there are  $1 \leq r, s \leq n$  such that  $Q_r \cap Q_s \subseteq I = \cap_{i=1}^n Q_i \subseteq Q_r \cap Q_s$ .  $\square$

Let  $S$  be a multiplicatively closed subset of a ring  $R$ . Set

$$C := \{I^c \mid I \text{ is an ideal of } R_S\}.$$

**Corollary 2.22.** *Let  $R$  be a ring and  $S$  be a multiplicatively closed subset of  $R$ . Then there is a one-to-one correspondence between the strongly 2-irreducible ideals of  $R_S$  and strongly 2-irreducible ideals of  $R$  contained in  $C$  which do not meet  $S$ .*

*Proof.* If  $I$  is a strongly 2-irreducible ideal of  $R_S$ , then evidently  $I^c \neq R$ ,  $I^c \in C$  and by Theorem 2.21(1),  $I^c$  is a strongly 2-irreducible ideal of  $R$ . Conversely, let  $I$  be a strongly 2-irreducible ideal of  $R$ ,  $I \cap S = \emptyset$  and  $I \in C$ . Since  $I \cap S = \emptyset$ ,  $I^c \neq R_S$ . Let  $J \cap K \cap L \subseteq I^c$  where  $J$ ,  $K$  and  $L$  are ideals of  $R_S$ . Then  $J^c \cap K^c \cap L^c = (J \cap K \cap L)^c \subseteq I^{ec}$ . Now since  $I \in C$ , then  $I^{ec} = I$ . So  $J^c \cap K^c \cap L^c \subseteq I$ . Hence, either  $J^c \cap K^c \subseteq I$  or  $J^c \cap L^c \subseteq I$  or  $K^c \cap L^c \subseteq I$ . Then, either  $J \cap K = (J \cap K)^{ce} \subseteq I^e$  or  $J \cap L = (J \cap L)^{ce} \subseteq I^e$  or  $K \cap L = (K \cap L)^{ce} \subseteq I^e$ . Consequently,  $I^e$  is a strongly 2-irreducible ideal of  $R_S$ .  $\square$

Let  $n$  be a natural number. We say that  $I$  is an  $n$ -primary ideal of a ring  $R$  if  $I$  is the intersection of  $n$  primary ideals of  $R$ .

**Proposition 2.23.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (1) *Every  $n$ -primary ideal of  $R$  is a strongly 2-irreducible ideal;*
- (2) *For any prime ideal  $P$  of  $R$ , every  $n$ -primary ideal of  $R_P$  is a strongly 2-irreducible ideal;*
- (3) *For any maximal ideal  $m$  of  $R$ , every  $n$ -primary ideal of  $R_m$  is a strongly 2-irreducible ideal.*

*Proof.* (1) $\Rightarrow$ (2) Let  $I$  be an  $n$ -primary ideal of  $R_P$ . We know that  $I^c$  is an  $n$ -primary ideal of  $R$ ,  $I^c \cap (R \setminus P) = \emptyset$ ,  $I^c \in C$  and, by the assumption,  $I^c$  is a strongly 2-irreducible ideal of  $R$ . Now, by Corollary 2.22,  $I = (I^c)_P$  is a strongly 2-irreducible ideal of  $R_P$ .

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let  $I$  be an  $n$ -primary ideal of  $R$  and let  $m$  be a maximal ideal of  $R$  containing  $I$ . Then,  $I_m$  is an  $n$ -primary ideal of  $R_m$  and so, by our assumption,  $I_m$  is a strongly 2-irreducible ideal of  $R_m$ . Now by Theorem 2.27(1),  $(I_m)^c$  is a strongly 2-irreducible ideal of  $R$ , and since  $I$  is an  $n$ -primary ideal of  $R$ ,  $(I_m)^c = I$ , that is,  $I$  is a strongly 2-irreducible ideal of  $R$ .  $\square$

**Theorem 2.24.** *Let  $R = R_1 \times R_2$ , where  $R_1$  and  $R_2$  are rings with  $1 \neq 0$ . Let  $J$  be a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  *$J$  is a strongly 2-irreducible ideal of  $R$ ;*
- (2) *Either  $J = I_1 \times R_2$  for some strongly 2-irreducible ideal  $I_1$  of  $R_1$  or  $J = R_1 \times I_2$  for some strongly 2-irreducible ideal  $I_2$  of  $R_2$  or  $J = I_1 \times I_2$  for some strongly irreducible ideal  $I_1$  of  $R_1$  and some strongly irreducible ideal  $I_2$  of  $R_2$ .*

*Proof.* (1) $\Rightarrow$ (2) Assume that  $J$  is a strongly 2-irreducible ideal of  $R$ . Then  $J = I_1 \times I_2$  for some ideal  $I_1$  of  $R_1$  and some ideal  $I_2$  of  $R_2$ . Suppose that  $I_2 = R_2$ . Since  $J$  is a proper ideal of  $R$ ,  $I_1 \neq R_1$ . Let  $R' = \frac{R}{\{0\} \times R_2}$ . Then  $J' = \frac{J}{\{0\} \times R_2}$  is a strongly 2-irreducible ideal of  $R'$  by Theorem 2.21(5). Since  $R'$  is ring-isomorphic to  $R_1$  and  $I_1 \simeq J'$ ,  $I_1$  is a strongly 2-irreducible ideal of  $R_1$ . Suppose that  $I_1 = R_1$ . Since  $J$  is a proper ideal of  $R$ ,  $I_2 \neq R_2$ . By a similar argument as in the previous case,  $I_2$  is a strongly 2-irreducible ideal of  $R_2$ . Hence assume that  $I_1 \neq R_1$  and  $I_2 \neq R_2$ . Suppose that  $I_1$  is not a strongly irreducible ideal of  $R_1$ . Then there are  $x, y \in R_1$  such that  $R_1x \cap R_1y \subseteq I_1$  but neither  $x \in I_1$  nor  $y \in I_1$ . Notice that  $(R_1x \times R_2) \cap (R_1 \times \{0\}) \cap (R_1y \times R_2) = (R_1x \cap R_1y) \times \{0\} \subseteq J$ , but neither  $(R_1x \times R_2) \cap (R_1 \times \{0\}) = R_1x \times \{0\} \subseteq J$  nor  $(R_1x \times R_2) \cap (R_1y \times R_2) = (R_1x \cap R_1y) \times R_2 \subseteq J$  nor  $(R_1 \times \{0\}) \cap (R_1y \times R_2) = R_1y \times \{0\} \subseteq J$ , which is a contradiction. Thus  $I_1$  is a strongly irreducible ideal of  $R_1$ . Suppose that  $I_2$  is not a strongly irreducible ideal of  $R_2$ . Then there are  $z, w \in R_2$  such that  $R_2z \cap R_2w \subseteq I_2$  but neither  $z \in I_2$  nor  $w \in I_2$ . Notice that  $(R_1 \times R_2z) \cap (\{0\} \times R_2) \cap (R_1 \times R_2w) = \{0\} \times (R_2z \cap R_2w) \subseteq J$ , but neither  $(R_1 \times R_2z) \cap (\{0\} \times R_2) = \{0\} \times R_2z \subseteq J$ , nor  $(R_1 \times R_2z) \cap (R_1 \times R_2w) = R_1 \times (R_2z \cap R_2w) \subseteq J$  nor  $(\{0\} \times R_2) \cap (R_1 \times R_2w) = \{0\} \times R_2w \subseteq J$ , which is a contradiction. Thus  $I_2$  is a strongly irreducible ideal of  $R_2$ .

(2) $\Rightarrow$ (1) If  $J = I_1 \times R_2$  for some strongly 2-irreducible ideal  $I_1$  of  $R_1$  or  $J = R_1 \times I_2$  for some strongly 2-irreducible ideal  $I_2$  of  $R_2$ , then it is clear that  $J$  is a strongly 2-irreducible ideal of  $R$ . Hence assume that  $J = I_1 \times I_2$  for some strongly irreducible ideal  $I_1$  of  $R_1$  and some strongly irreducible ideal  $I_2$  of  $R_2$ . Then  $I'_1 = I_1 \times R_2$  and  $I'_2 = R_1 \times I_2$  are strongly irreducible ideals of  $R$ . Hence  $I'_1 \cap I'_2 = I_1 \times I_2 = J$  is a strongly 2-irreducible ideal of  $R$  by Proposition 2.13.  $\square$

**Theorem 2.25.** *Let  $R = R_1 \times R_2 \times \cdots \times R_n$ , where  $2 \leq n < \infty$ , and  $R_1, R_2, \dots, R_n$  are rings with  $1 \neq 0$ . Let  $J$  be a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  *$J$  is a strongly 2-irreducible ideal of  $R$ .*
- (2) *Either  $J = \times_{t=1}^n I_t$  such that for some  $k \in \{1, 2, \dots, n\}$ ,  $I_k$  is a strongly 2-irreducible ideal of  $R_k$ , and  $I_t = R_t$  for every  $t \in \{1, 2, \dots, n\} \setminus \{k\}$  or  $J = \times_{t=1}^n I_t$  such that for some  $k, m \in \{1, 2, \dots, n\}$ ,  $I_k$  is a strongly irreducible ideal of  $R_k$ ,  $I_m$  is a strongly irreducible ideal of  $R_m$ , and  $I_t = R_t$  for every  $t \in \{1, 2, \dots, n\} \setminus \{k, m\}$ .*

*Proof.* We use induction on  $n$ . Assume that  $n = 2$ . Then the result is valid by Theorem 2.24. Thus let  $3 \leq n < \infty$  and assume that the result is valid when  $K = R_1 \times \cdots \times R_{n-1}$ . We prove the result when  $R = K \times R_n$ . By Theorem 2.24,  $J$  is a strongly 2-irreducible ideal of  $R$  if and only if either  $J = L \times R_n$  for some strongly 2-irreducible ideal  $L$  of  $K$  or  $J = K \times L_n$  for some strongly 2-irreducible ideal  $L_n$  of  $R_n$  or  $J = L \times L_n$  for some strongly irreducible ideal  $L$  of  $K$  and some strongly irreducible ideal  $L_n$  of  $R_n$ . Observe that a proper ideal  $Q$  of  $K$  is a strongly irreducible ideal of  $K$  if and only if  $Q = \times_{t=1}^{n-1} I_t$  such that for some  $k \in \{1, 2, \dots, n-1\}$ ,  $I_k$  is a strongly irreducible ideal of  $R_k$ , and  $I_t = R_t$  for every  $t \in \{1, 2, \dots, n-1\} \setminus \{k\}$ . Thus the claim is now verified.  $\square$

**Lemma 2.26.** *Let  $R$  be a GCD-domain and  $I$  be a proper ideal of  $R$ . The following conditions are equivalent:*

- (1)  *$I$  is a singly strongly 2-irreducible ideal;*
- (2) *For every elements  $x, y, z \in R$ ,  $[x, y, z] \in I$  implies that  $[x, y] \in I$  or  $[x, z] \in I$  or  $[y, z] \in I$ .*

*Proof.* Since for every elements  $x, y$  of  $R$  we have  $Rx \cap Ry = R[x, y]$ , there is nothing to prove.  $\square$

Now we study singly strongly 2-irreducible ideals of a  $UFD$ .

**Theorem 2.27.** *Let  $R$  be a  $UFD$ , and let  $I$  be a proper ideal of  $R$ . Then the following conditions hold:*

- (1)  *$I$  is singly strongly 2-irreducible if and only if for each elements  $x, y, z$  of  $R$ ,  $[x, y, z] \in I$  implies that either  $[x, y] \in I$  or  $[x, z] \in I$  or  $[y, z] \in I$ .*
- (2)  *$I$  is singly strongly 2-irreducible if and only if  $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$ , where  $p_i$ 's are distinct prime elements of  $R$  and  $n_i$ 's are natural numbers, implies that  $p_r^{n_r} p_s^{n_s} \in I$ , for some  $1 \leq r, s \leq k$ .*
- (3) *If  $I$  is a nonzero principal ideal, then  $I$  is singly strongly 2-irreducible if and only if the generator of  $I$  is a prime power or the product of two prime powers.*
- (4) *Every singly strongly 2-irreducible ideal is a 2-absorbing primary ideal.*

*Proof.* (1) By Lemma 2.26.

(2) Suppose that  $I$  is singly strongly 2-irreducible and  $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$  in which  $p_i$ 's are distinct prime elements of  $R$  and  $n_i$ 's are natural numbers. Then  $[p_1^{n_1}, p_2^{n_2}, \dots, p_k^{n_k}] = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$ . Hence by part (1), there are  $1 \leq r, s \leq k$  such that  $[p_r^{n_r}, p_s^{n_s}] \in I$ , i.e.,  $p_r^{n_r} p_s^{n_s} \in I$ .

For the converse, let  $[x, y, z] \in I$  for some  $x, y, z \in R \setminus \{0\}$ . Assume that  $x, y$

and  $z$  have prime decompositions as below,

$$\begin{aligned} x &= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}, \\ y &= p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k} r_1^{\delta_1} r_2^{\delta_2} \cdots r_u^{\delta_u}, \\ z &= p_1^{\varepsilon_1} p_2^{\varepsilon_2} \cdots p_{k'}^{\varepsilon_{k'}} q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_{s'}^{\lambda_{s'}} r_1^{\mu_1} r_2^{\mu_2} \cdots r_{u'}^{\mu_{u'}} s_1^{\kappa_1} s_2^{\kappa_2} \cdots s_v^{\kappa_v}, \end{aligned}$$

in which  $0 \leq k' \leq k$ ,  $0 \leq s' \leq s$  and  $0 \leq u' \leq u$ . Therefore,

$$\begin{aligned} [x, y, z] &= p_1^{\nu_1} p_2^{\nu_2} \cdots p_{k'}^{\nu_{k'}} p_{k'+1}^{\omega_{k'+1}} \cdots p_k^{\omega_k} q_1^{\rho_1} q_2^{\rho_2} \cdots q_{s'}^{\rho_{s'}} \\ &\quad q_{s'+1}^{\beta_{s'+1}} \cdots q_s^{\beta_s} r_1^{\sigma_1} r_2^{\sigma_2} \cdots r_{u'}^{\sigma_{u'}} r_{u'+1}^{\delta_{u'+1}} \cdots r_u^{\delta_u} s_1^{\kappa_1} s_2^{\kappa_2} \cdots s_v^{\kappa_v} \in I, \end{aligned}$$

where  $\nu_i = \max\{\alpha_i, \gamma_i, \varepsilon_i\}$  for every  $1 \leq i \leq k'$ ;  $\omega_j = \max\{\alpha_j, \gamma_j\}$  for every  $k' < j \leq k$ ;  $\rho_i = \max\{\beta_i, \lambda_i\}$  for every  $1 \leq i \leq s'$ ;  $\sigma_i = \max\{\delta_i, \mu_i\}$  for every  $1 \leq i \leq u'$ . By part (2), we have twenty one cases. For example we investigate the following two cases. The other cases can be verified in a similar way.

**Case 1.** For some  $1 \leq i, j \leq k'$ ,  $p_i^{\nu_i} p_j^{\nu_j} \in I$ . If  $\nu_i = \alpha_i$  and  $\nu_j = \alpha_j$ , then clearly  $x \in I$  and so  $[x, y] \in I$ . If  $\nu_i = \alpha_i$  and  $\nu_j = \gamma_j$ , then  $p_i^{\alpha_i} p_j^{\gamma_j} \mid [x, y]$  and thus  $[x, y] \in I$ . If  $\nu_i = \alpha_i$  and  $\nu_j = \varepsilon_j$ , then  $p_i^{\alpha_i} p_j^{\varepsilon_j} \mid [x, z]$  and thus  $[x, z] \in I$ .

**Case 2.** Let  $p_i^{\nu_i} p_j^{\omega_j} \in I$ ; for some  $1 \leq i \leq k'$  and  $k' + 1 \leq j \leq k$ . For  $\nu_i = \alpha_i$ ,  $\omega_j = \alpha_j$  we have  $x \in I$  and so  $[x, y] \in I$ . For  $\nu_i = \varepsilon_i$ ,  $\omega_j = \gamma_j$  we have  $[y, z] \in I$ .

Consequently  $I$  is singly strongly 2-irreducible, by part (1).

(3) Suppose that  $I = Ra$  for some nonzero element  $a \in R$ . Assume that  $I$  is singly strongly 2-irreducible. Let  $a = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$  be a prime decomposition for  $a$  such that  $k > 2$ . By part (2) we have that  $p_r^{n_r} p_s^{n_s} \in I$  for some  $1 \leq r, s \leq k$ . Therefore  $I = R(p_r^{n_r} p_s^{n_s})$ .

Conversely, if  $a$  is a prime power, then  $I$  is strongly irreducible ideal, by [3, Theorem 2.2(3)]. Hence  $I$  is singly strongly 2-irreducible. Let  $I = R(p^r q^s)$  for some prime elements  $p, q$  of  $R$ . Assume that for some distinct prime elements  $q_1, q_2, \dots, q_k$  of  $R$  and natural numbers  $m_1, m_2, \dots, m_k$ ,  $q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k} \in I = R(p^r q^s)$ . Then  $p^r q^s \mid q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}$ . Hence there exists  $1 \leq i \leq k$  such that  $p = q_i$  and  $r \leq m_i$ , also there exists  $1 \leq j \leq k$  such that  $q = q_j$  and  $s \leq m_j$ . Then, since  $p^r q^s \in I$ , we have  $q_i^{m_i} q_j^{m_j} \in I$ . Now, by part (2),  $I$  is singly strongly 2-irreducible.

(4) Let  $I$  be singly strongly 2-irreducible and  $xyz \in I$  for some  $x, y, z \in R \setminus \{0\}$ . Consider the following prime decompositions,

$$\begin{aligned} x &= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}, \\ y &= p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k} r_1^{\delta_1} r_2^{\delta_2} \cdots r_u^{\delta_u}, \\ z &= p_1^{\varepsilon_1} p_2^{\varepsilon_2} \cdots p_{k'}^{\varepsilon_{k'}} q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_{s'}^{\lambda_{s'}} r_1^{\mu_1} r_2^{\mu_2} \cdots r_{u'}^{\mu_{u'}} s_1^{\kappa_1} s_2^{\kappa_2} \cdots s_v^{\kappa_v}, \end{aligned}$$

in which  $0 \leq k' \leq k$ ,  $0 \leq s' \leq s$  and  $0 \leq u' \leq u$ . By these representations we have,

$$\begin{aligned}
xyz &= p_1^{\alpha_1+\gamma_1+\varepsilon_1} p_2^{\alpha_2+\gamma_2+\varepsilon_2} \cdots p_{k'}^{\alpha_{k'}+\gamma_{k'}+\varepsilon_{k'}} p_{k'+1}^{\alpha_{k'+1}+\gamma_{k'+1}} \\
&\quad \cdots p_k^{\alpha_k+\gamma_k} q_1^{\beta_1+\lambda_1} q_2^{\beta_2+\lambda_2} \cdots q_{s'}^{\beta_{s'}+\lambda_{s'}} q_{s'+1}^{\beta_{s'+1}} \cdots q_s^{\beta_s} \\
&\quad r_1^{\delta_1+\mu_1} r_2^{\delta_2+\mu_2} \cdots r_{u'}^{\delta_{u'}+\mu_{u'}} r_{u'+1}^{\delta_{u'+1}} \cdots r_u^{\delta_u} s_1^{\kappa_1} s_2^{\kappa_2} \cdots s_v^{\kappa_v} \in I.
\end{aligned}$$

Now, apply part (2). We investigate some cases that can be happened, the other cases similarly lead us to the claim that  $I$  is 2-absorbing primary. First, assume for some  $1 \leq i, j \leq k'$ ,  $p_i^{\alpha_i+\gamma_i+\varepsilon_i} p_j^{\alpha_j+\gamma_j+\varepsilon_j} \in I$ . Choose a natural number  $n$  such that  $n \geq \max\{\frac{\alpha_i+\gamma_i}{\varepsilon_i}, \frac{\alpha_j+\gamma_j}{\varepsilon_j}\}$ . With this choice we have  $(n+1)\varepsilon_i \geq \alpha_i+\gamma_i+\varepsilon_i$  and  $(n+1)\varepsilon_j \geq \alpha_j+\gamma_j+\varepsilon_j$ , so  $p_i^{(n+1)\varepsilon_i} p_j^{(n+1)\varepsilon_j} \in I$ . Then  $z^{n+1} \in I$ , so  $z \in \sqrt{I}$ . The other one case; assume that for some  $1 \leq i \leq k'$  and  $k'+1 \leq j \leq k$ ,  $p_i^{\alpha_i+\gamma_i+\varepsilon_i} p_j^{\alpha_j+\gamma_j} \in I$ . Choose a natural number  $n$  such that  $n \geq \max\{\frac{\alpha_i+\varepsilon_i}{\gamma_i}, \frac{\alpha_j}{\gamma_j}\}$ . With this choice we have  $(n+1)\gamma_i \geq \alpha_i+\gamma_i+\varepsilon_i$  and  $(n+1)\gamma_j \geq \alpha_j+\gamma_j$ , thus  $p_i^{(n+1)\gamma_i} p_j^{(n+1)\gamma_j} \in I$ . Then  $y^{n+1} \in I$ , so  $y \in \sqrt{I}$ . Assume that  $p_i^{\alpha_i+\gamma_i} s_j^{\kappa_j} \in I$ , for some  $k'+1 \leq i \leq k$  and some  $1 \leq j \leq v$ . Let  $n$  be a natural number where  $n \geq \frac{\gamma_i}{\alpha_i}$ , then  $(n+1)\alpha_i \geq \alpha_i+\gamma_i$ . Hence  $p_i^{(n+1)\alpha_i} s_j^{(n+1)\kappa_j} \in I$  which shows that  $xz \in \sqrt{I}$ . Suppose that for some  $s'+1 \leq i \leq s$  and  $u'+1 \leq j \leq u$ ,  $q_i^{\beta_i} r_j^{\delta_j} \in I$ . Then, clearly  $xy \in I$ .  $\square$

**Corollary 2.28.** *Let  $R$  be a UFD.*

- (1) *Every principal ideal of  $R$  is a singly strongly 2-irreducible ideal if and only if it is a 2-absorbing primary ideal.*
- (2) *Every singly strongly 2-irreducible ideal of  $R$  can be generated by a set of elements of the forms  $p^n$  and  $p_i^{n_i} p_j^{n_j}$  in which  $p, p_i, p_j$  are some prime elements of  $R$  and  $n, n_i, n_j$  are some natural numbers.*
- (3) *Every 2-absorbing ideal of  $R$  is a singly strongly 2-irreducible ideal.*

*Proof.* (1) Suppose that  $I$  is singly strongly 2-irreducible ideal. By Theorem 2.27(4),  $I$  is a 2-absorbing primary ideal. Conversely, let  $I$  be a nonzero 2-absorbing primary ideal. Let  $I = Ra$ , where  $0 \neq a \in I$ . Assume that  $a = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$  be a prime decomposition for  $a$ . If  $k > 2$ , then since  $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$  and  $I$  is a 2-absorbing primary ideal, there exist a natural number  $n$ , and integers  $1 \leq i, j \leq k$  such that  $p_i^{nn_i} p_j^{nn_j} \in I$ , say  $i = 1$  and  $j = 2$ . Therefore  $p_3 \mid p_1^{nn_1} p_2^{nn_2}$  which is a contradiction. Therefore  $k = 1$  or 2, that is  $I = Rp_1^{n_1}$  or  $I = R(p_1^{n_1} p_2^{n_2})$ , respectively. Hence by Theorem 2.27(3),  $I$  is singly strongly 2-irreducible.

(2) Let  $X$  be a generator set for a singly strongly 2-irreducible ideal of  $I$ , and let  $x$  be a nonzero element of  $X$ . Assume that  $x = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$  be a prime decomposition for  $x$  such that  $k \geq 2$ . By Theorem 2.27(2), for some  $1 \leq i, j \leq k$ , we have  $p_i^{n_i} p_j^{n_j} \in I$ , and then  $Rx \subseteq Rp_i^{n_i} p_j^{n_j} \subseteq I$ . Consequently,  $I$  can be generated by a set of elements of the forms  $p^n$  and  $p_i^{n_i} p_j^{n_j}$ .

(3) is a direct consequence of Theorem 2.27(2).  $\square$

The following example shows that in part (1) of Corollary 2.28 the condition that  $I$  is principal is necessary. Moreover, the converse of part (2) of this corollary need not be true.

**Example 2.29.** Let  $F$  be a field and  $R = F[x, y, z]$ , where  $x$ ,  $y$  and  $z$  are independent indeterminates. We know that  $R$  is a *UFD*. Suppose that  $I = \langle x, y^2, z^2 \rangle$ . Since  $\sqrt{\langle x, y^2, z^2 \rangle} = \langle x, y, z \rangle$  is a maximal ideal of  $R$ ,  $I$  is a primary ideal and so is a 2-absorbing primary ideal. Notice that  $(x + y + z)yz \in I$ , but neither  $(x + y + z)y \in I$  nor  $(x + y + z)z \in I$  nor  $yz \in I$ . Consequently,  $I$  is not singly strongly 2-irreducible, by Theorem 2.27(2).

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